

Contravariant, Covariant, and Mixed Tensors

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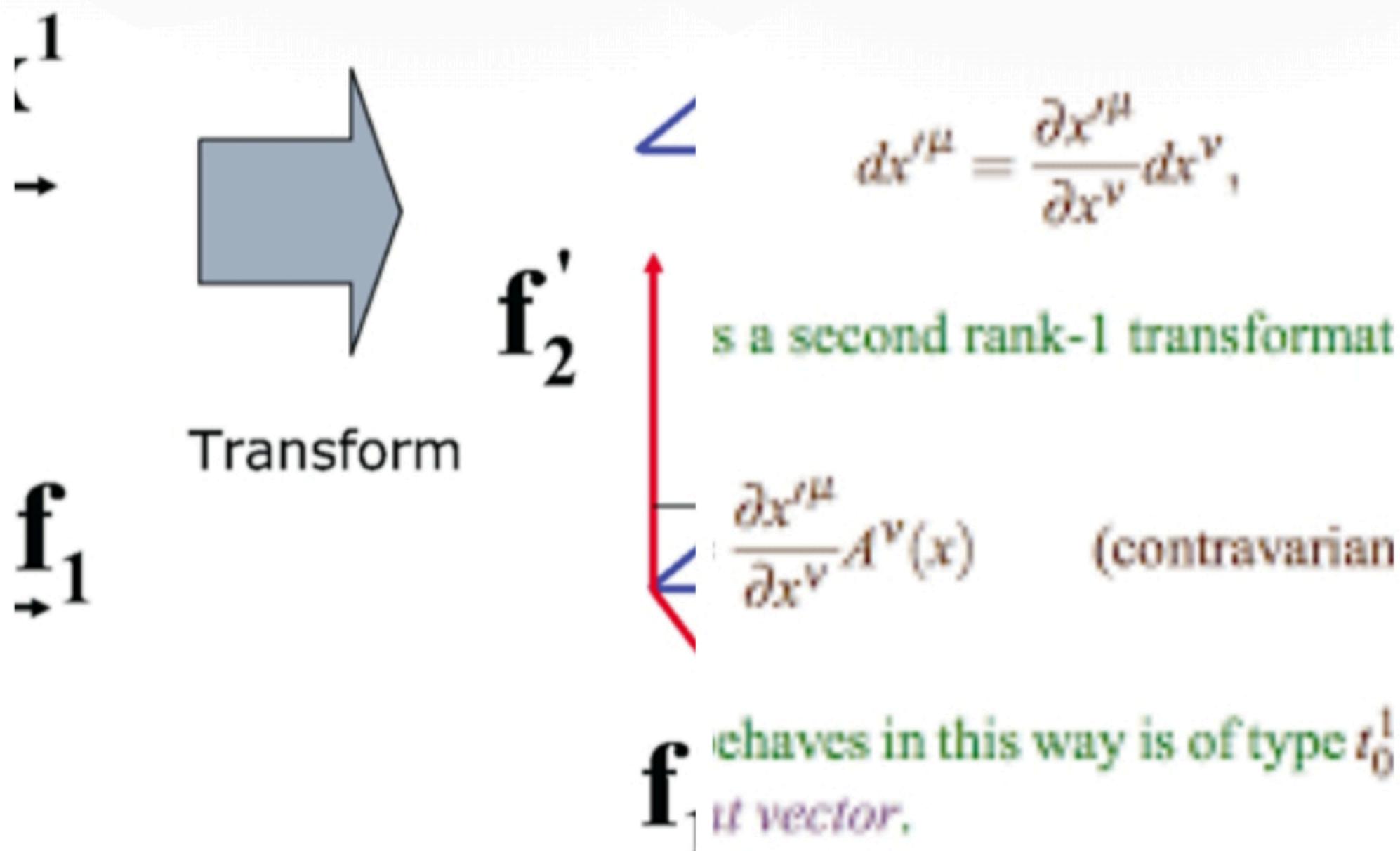
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1. Introduction to Tensors

In mathematical physics, a **tensor** is a physical or mathematical quantity whose components transform according to definite laws under a change of coordinates. Scalars, vectors, and matrices are special cases of tensors.

Tensor analysis provides a coordinate-independent formulation of physical laws and is extensively used in classical mechanics, electromagnetism, fluid mechanics, and relativity.

2. Contravariant Tensor



Definition

A **contravariant tensor** is a tensor whose components transform **inversely** with respect to a change of coordinates.

Notation

Contravariant components are represented by **upper indices**:

$$A^i$$

Transformation Law

If the coordinate system changes from x^i to x'^i , then the contravariant components transform as:

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

Examples

- Position vector
- Displacement vector
- Velocity vector

Physical Significance

Contravariant components follow the coordinate directions and change inversely when the scale of coordinates is altered.

3. Covariant Tensor

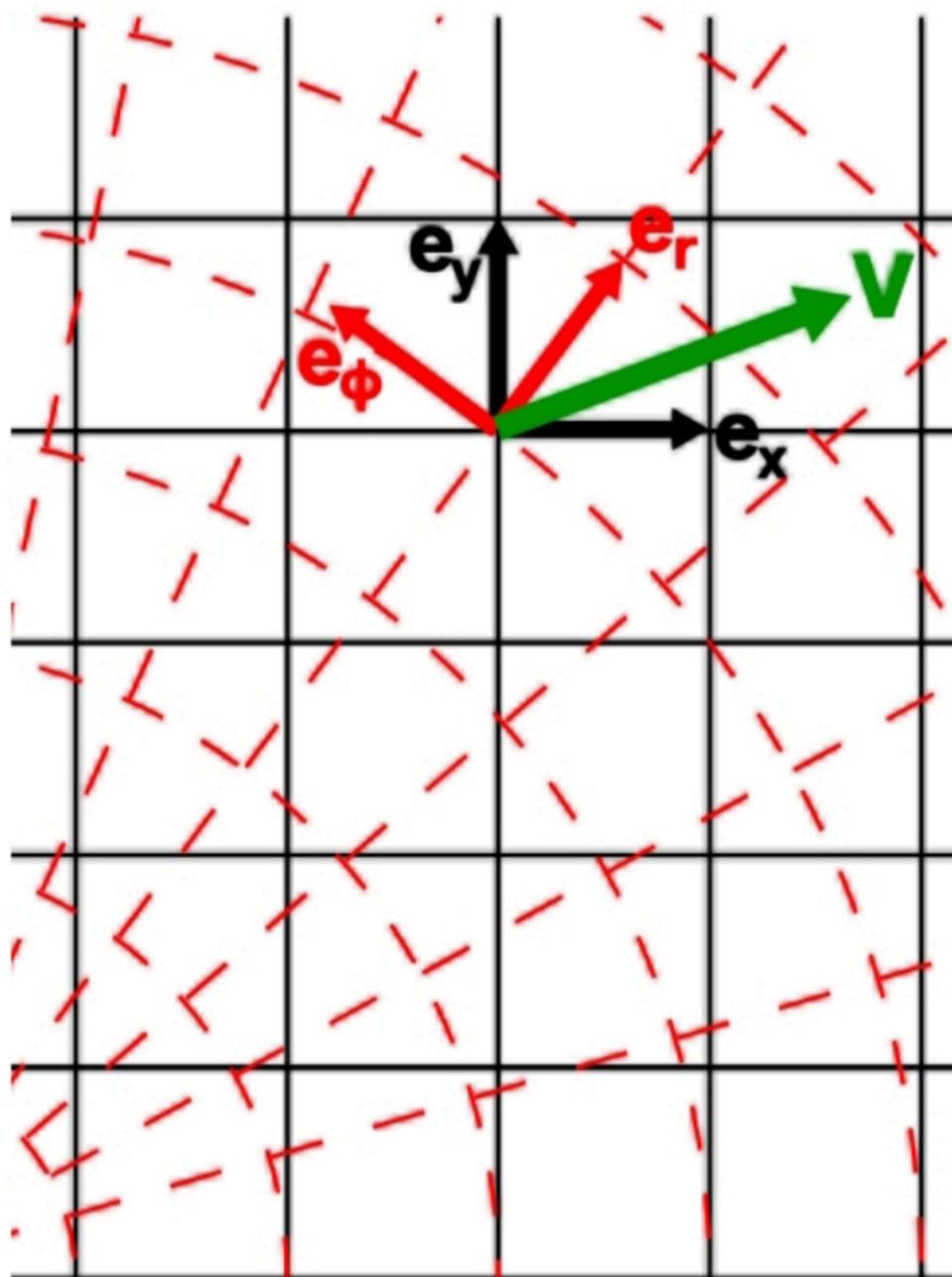
COVARIANT VECTORS: A diffeomorphism

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu},$$

is a second rank-1 transformation

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}(x) \quad (\text{contravariant})$$

changes in this way is of type t_0^1 if vector.



Definition

A **covariant tensor** is a tensor whose components transform **directly** with the change of coordinates.

Notation

Covariant components are written with **lower indices**:

$$A_i$$

Transformation Law

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j$$

Examples

- Gradient of a scalar field
- Differential displacement
- Covariant form of force

Physical Significance

Covariant components are associated with surfaces of constant coordinates and often arise from gradients of scalar quantities.

4. Mixed (Contravariant–Covariant) Tensor

Tensors

It can be defined such as

$$T^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} X^d_{ef}$$

Contravariant rank 1 and covariant rank 2
An example of a mixed rank tensor is the
stress tensor, which relates the α^{th} component of the 3-
vector of the electric field \mathbf{E} .

$$E_g; \quad j_x = \sigma_x^x E_x + \sigma_x^y E_y$$

<http://www.physics.usyd.edu.au/~gfr/>

Definition

A tensor having **both contravariant and covariant indices** is called a **mixed tensor**.

$\vec{v}, \underline{\quad}$

$\vec{v}, \underline{\quad}$

Notation

$$T^i_j$$

Transformation Law

$$T'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} T^k_l$$

Examples

- Stress tensor
- Moment of inertia tensor
- Energy–momentum tensor

5. Raising and Lowering of Indices

The **metric tensor** is used to convert contravariant components into covariant components and vice versa.

Lowering an Index

$$A_i = g_{ij} A^j$$

Raising an Index

$$A^i = g^{ij} A_j$$

6. Comparison Between Contravariant and Covariant Tensors

| Feature | Contravariant Tensor |
|---------------------|-----------------------------|
| Index position | Upper index (\wedge) |
| Transformation rule | Inverse |
| Example | Velocity |
| Relation to metric | Fundamental |

7. Conclusion

Contravariant, covariant, and mixed tensors form the foundation of tensor analysis. Their transformation laws ensure that physical equations remain invariant under coordinate transformations. These concepts are essential for understanding advanced topics in **B.Sc. Mathematical Physics**.